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Logic of Likelihood

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The notion of "probability" is generalized to that of "likelihood," and a natural logical structure is shown to exist for any physical theory which predicts likelihoods. Two physically based axioms are given for this logical structure to form an orthomodular poset, with an order-determining set of states. The results strengthen the basis of the quantum logic approach to axiomatic quantum theory.

1. INTRODUCTION

It has long been noted that every theory which makes statistical predictions has an associated probability and logical structure (Husimi, 1937; Maczynski, 1974; Hall, 1988). Moreover, it has been shown that the well-known Bell inequalities provide necessary conditions for such a theory to both (a) have a *classical* structure (i.e., a Boolean logic with Kolmogorovian probabilities); and (b) satisfy a weak statistical-locality condition (Hall, 1988).

The apparent violation of the Bell inequalities in nature (Aspect *et al.*, 1982) strongly suggests then that a local description of physical phenomena may be retained *only* by restriction to theories with *nonclassical* probability and logical structures. This provides a new, semiempirical motivation for the study of such structures in physics.

In fact, motivated initially by an analogy between the formalisms of classical and quantum mechanics (Birkhoff and von Neumann, 1936), such structures have long been studied within the *quantum logic approach* to axiomatic quantum theory (e.g., Hooker, 1979). The mathematical nucleus of this approach is formed by *orthomodular posets*, which neatly extract and generalize essential "quantum" features of the standard Hilbert space formalism (e.g., Beltrametti and Cassinelli, 1981; Varadarajan, 1985).

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A primary aim of the present paper is to augment the foundations of the quantum logic approach, by providing two physically based conditions for the inherent logical structure of a statistical theory to form an orthomodular poset. However, the result is obtained within a more general setting, in which the concept of "probability" is generalized to that of "likelihood."

In particular, whereas *probabilities* take values on the involutive poset $([0, 1], \leq, \sim)$, where $\leq$ is the natural ordering of the reals and $\sim p = 1 - p$, *likelihoods* (defined in Section 3) may take values on *any* involutive poset. It is shown in Section 4 that any theory which predicts likelihoods has a natural, "poset-valued" logical structure, called the predictive logic of the theory. Under two simple axioms, this predictive logic is shown to be an orthomodular poset (Section 5).

There are several conceptual advantages gained in replacing probability by the more abstract notion of likelihood:

(a) The fundamental nature of orthomodular posets is emphasized by the results, which are independent of particular features of probability, such as additivity and well-ordering.
(b) The results indicate that probability, with its various interpretational difficulties, need not be accepted as an *a priori* nonanalyzable component in axiomatic approaches to quantum mechanics.
(c) A particular form of likelihood, corresponding to the 3-valued or ternary lattice (Section 3), is of particular relevance to the quantum logic approach (Section 5).

A general characterization of physical theories in the following section provides the basic framework for discussing likelihood in Section 3, and logical structures in Sections 4 and 5. Dynamical aspects are considered in Section 6, indicating the primacy of the Heisenberg picture over the Schrödinger picture within the framework of likelihood theories. Conclusions are presented in Section 7.

2. CHARACTERIZATION OF PHYSICAL THEORIES

A theory for a class of physical systems will in general have both a *descriptive* and a *predictive* component. In this section notation is developed to enable discussion of those theories which predict "likelihoods" (and in particular those theories which predict probabilities).

It will be assumed here that the descriptive component of the theory is made up of a triple (S, X, P), where S denotes a set of *states*, describing individual members of the class, X denotes a set of *experiments*, describing physical operations which may be performed on any member of the class, and P denotes a set of *experimental propositions*, describing the possible outcomes of experiments in X (see further below).

For various reasons it may appear attractive to assume that S and X are *a priori* structurally related. For example, states might be identified with experiments that have single outcomes (i.e., "preparation procedures"), or indeed each experimental outcome might be identified with some state (regarding measuring devices as "filters"). Alternatively, Foulis *et al.* (1983) and Bennett and Foulis (1990) consider the case for which (S, X) forms a so-called "entity," where the elements of S correspond to subsets of experimental outcomes satisfying certain intersection conditions (Example 3.2 below). It will not be necessary in this paper to assume *any* prior relationships between S and X.

For each experiment $E \in X$, let $R(E)$ denote the set of possible results for E. Then, for $\alpha \subseteq R(E)$, let E_α denote the *experimental proposition*, "The result of E is contained in subset α of $R(E)$." Thus, E_α is either verified or falsified by each performance of experiment E. The set of experimental propositions of the theory will be denoted by P, as indicated earlier.

Since the totality of experimental propositions $\{E_\alpha\}$ for a given E characterizes the possible results of experiment E, the role of the predictive component of the theory may be formulated as follows: *to provide prior information concerning the verification/falsification of various experimental propositions in P, when tested for systems described by various states in S.*

For example, a *statistical* theory may be characterized as containing a mapping p from $P \times S$ to the interval $[0, 1]$, such that $p(E_\alpha, s)$ is predicted as *the probability that proposition $E_\alpha \in P$ will be verified if tested for state $s \in S$.* To qualify for such an interpretation, p must of course satisfy the conditions

$$p(E_{R(E)}, s) = 1 \tag{1a}$$

$$p(E_{\alpha \cup \beta}, s) = p(E_\alpha, s) + p(E_\beta, s) \qquad \text{for} \quad \alpha \cap \beta = \varnothing \tag{1b}$$

for every $E \in X$, $s \in S$ [note that only *finite* additivity is specified in (1b), since any physical experiment can only distinguish among a finite number of possible results]. A *deterministic* theory is defined as a special case of a statistical theory, with $p(E_\alpha, s)$ restricted to the range $\{0, 1\}$. The inherent logical structures of such theories, which include both classical and quantum mechanics, have been discussed in Hall (1988). It will be seen in Section 4 that such structures exist for any theory which makes poset-valued predictions.

3. CHARACTERIZATION OF LIKELIHOOD THEORIES

3.1. Basic Postulates

Generalizing from the case of statistical theories, a *likelihood theory* may be provisionally defined as a physical theory with descriptive component (S, X, P) and predictive component (L, l), where L is a set of *likelihoods*, l

is a mapping from $P \times S$ to L, and $l(E_\alpha, s)$ is predicted as *the likelihood that experimental proposition $E_\alpha \in P$ will be verified, if tested for state $s \in S$.*

To capture intuitive notions of likelihood, certain conditions should be imposed on l and L. For example, it might be required that L be the interval [0, 1], and that equations (1) be satisfied with p replaced by l. In this case "probability" and "likelihood" are equivalent. However, as discussed in the Introduction (see also Section 3.3), there are some advantages to be gained by imposing less restrictive requirements at this stage. Three fundamental requirements are postulated below.

First, for an experiment $E \in X$ with result set $R(E)$ (see previous section), let α^c denote the relative complement $R(E) \backslash \alpha$ for each $\alpha \subseteq R(E)$. Thus, the experimental proposition E_{α^c} is verified by experiment E if proposition E_α is falsified, and vice versa. It follows that any prior information concerning the verification/falsification of E_{α^c} stands in a one–one relation with information concerning the verification/falsification of E_α, motivating the following postulate:

(L1) The likelihood of experimental proposition E_α being verified for a given state determines the likelihood of experimental proposition E_{α^c} being verified for that state.

Postulate (L1) is essentially equivalent to Axiom $(1 \cdot \mathrm{i})$ of Cox (1961) in his axiomatization of probability, and implies that there exists an *involution* $\sim$ on L, such that for all $E \in X$, $s \in S$,

$$l(E_{\alpha^c}, s) = \sim l(E_\alpha, s) \tag{2a}$$

Second, for an experiment $E \in X$ consider two experimental propositions E_α, $E_\beta \in P$ with $\alpha \subseteq \beta$. Thus E_β will be verified by experiment E whenever E_α is verified, and may possibly be verified even if E_α is falsified. In an intuitive sense, then, the verification of E_α is *less likely* than the verification of E_β, suggesting that certain likelihoods should be comparable, or *ordered.* This motivates:

(L2) There exists a partial ordering $\leq$ on L such that for all $E \in X$, $s \in S$,

$$l(E_\alpha, s) \leq l(E_\beta, s) \qquad \text{if} \quad \alpha \subseteq \beta \tag{2b}$$

Note here that a binary relation $\leq$ on L is a partial ordering if for all $x, y, z \in L$ one has: $x \leq x$ (reflexivity); if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity); and if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry). Postulate (L2) is essentially a generalization of Axioms 1 and 2 of Jeffreys (1961) in his axiomatization of probability, which further require the relation $\leq$ to be a *total* ordering (i.e., $x \leq y$ or $y \leq x$ for all $x, y \in L$).

Third and finally, for two experiments $E, F \in X$ consider the experimental propositions $E_\varnothing, F_\varnothing \in P$, where $\varnothing$ denotes the empty set. These propositions are *a priori* always falsified by experiment, and hence intuitively should be equivalent with respect to their "zero" likelihood of verification. This motivates the normative postulate:

(L3) The likelihoods of any two "absurd" propositions $E_\varnothing, F_\varnothing$ being verified for a given state are equal, i.e., for all $E, F \in X, s \in S$,

$$l(E_\varnothing, s) = l(F_\varnothing, s) \tag{2c}$$

Postulate (L3) is related to Axiom 3 of Jeffreys (1961), which in part states that absurd propositions have equal probabilities.

Postulates (L1)–(L3), or equivalently conditions (2a)–(2c), provide sufficient restrictions on the notion of likelihood for deriving the results of this paper. Any triple $(L, \leq, \sim)$, where $(L, \leq)$ forms a partially ordered set and $\sim$ is an involution on L, will be called a *likelihood poset*. The provisional definition of likelihood theories given at the beginning of this section may now be made more precise:

Definition. A likelihood theory is a physical theory with descriptive component (S, X, P), and predictive component $(\mathscr{L}, l)$, where $\mathscr{L} = (L, \leq, \sim)$ is a likelihood poset, and l is a mapping from $P \times S$ to L which satisfies conditions (2a)–(2c).

3.2. Examples of Likelihood

Some important examples of likelihood posets, and the nature of their corresponding likelihood theories, are briefly discussed below.

Example 3.1. The binary lattice and deterministic theories. Likelihood theories which predict the same values for $l(E_\varnothing, s)$ and $l(E_{R(E)}, s)$ are trivial. Among nontrivial theories, the simplest likelihood poset is the *binary lattice* $\mathscr{L}_2 = (L_2, \leq, \sim)$, where L_2 has exactly two elements, interchanged under involution. Two suggestive notations for L_2 are $\{F, T\}$ and $\{0, 1\}$; the latter will be adopted here. For a *binary* likelihood theory, with predictive component $(\mathscr{L}_2, l)$, it can be checked that the requirement $l(E_\varnothing, s) = 0$ merely removes the interchange symmetry between the labels 0 and 1, and so may be adopted as a convention. With this convention it follows that $l(E_{R(E)}, s) = 1$, from condition (2a), and hence that $0 \leq 1$ from condition (2b).

In general, the binary likelihood values 0 and 1 may be interpreted as "unlikely" and "likely," respectively. The stronger interpretation of 0 as "impossible" [and hence 1 as "certain" via (2a)] is consistent only if the condition

$$l(E_\alpha, s) = 0 \qquad \text{implies} \qquad l(E_{\alpha\cup\beta}, s) \le l(E_\beta, s) \tag{3}$$

is satisfied. This condition ensures that a result $r \in \alpha \cup \beta$ is no more likely than a result $r \in \beta$, whenever a result $r \in \alpha$ is "impossible." Condition (3) is of course satisfied by all *deterministic* theories (defined at the end of Section 2), where the prediction $l(E_\alpha, s) = 0$ corresponds to the impossibility of experimental proposition E_α being verified, if tested for state s.

Example 3.2. The ternary lattice and "entities." The simplest nontrivial likelihood poset, after the binary lattice, is the *ternary lattice* $\mathscr{L}_3 = (L_3, \le, \sim)$, where $L_3 = \{0, \frac{1}{2}, 1\}$ has exactly three elements, satisfying $\sim 0 = 1$, $\sim\frac{1}{2} = \frac{1}{2}$, and $0 \le \frac{1}{2} \le 1$. If condition (3) is satisfied, then the likelihood values $0, \frac{1}{2}, 1$ can be consistently interpreted as corresponding to the notions "impossible," "indeterminate," and "certain," which are natural precursors to the concept of probability.

The class of *ternary* likelihood theories, with predictive component $(\mathscr{L}_3, l)$, is related to the "entities" defined by Foulis *et al.* (1983; see also Bennett and Foulis, 1990). In the notation of the present paper, the descriptive component (S, X, P) of a physical theory forms an *entity* if the elements of S correspond to nonempty subsets of experimental outcomes, such that both the "covering condition"

$$\bigcup_{s\in S} s = \bigcup_{E\in X} R(E) \tag{4a}$$

and the "exchange condition"

$$s \cap R(E) \subseteq R(F) \qquad \text{implies} \qquad s \cap R(F) \subseteq R(E) \tag{4b}$$

are satisfied. According to Bennett and Foulis (1990, p. 735), "the outcomes that belong to a state s are understood to be those that could be obtained . . . when the entity is in state s," i.e., the outcomes which are not in s are "precisely those that are *impossible* in state s." Hence an entity corresponds naturally to a ternary likelihood theory, with the mapping l_3 from $P \times S$ to L_3 defined by

$$l_3(E_\alpha, s) := \begin{cases} 0, & \alpha \cap s = \varnothing \\ 1, & \alpha^c \cap s = \varnothing \\ \frac{1}{2}, & \text{otherwise} \end{cases} \tag{5}$$

It can be checked that conditions (2a)–(2c) and (3) are satisfied by this definition [conditions (4) imply $s \cap R(E) \neq \varnothing$ always, and hence that l_3 is indeed well-defined]. Entities can therefore be characterized as corresponding to a special class of ternary likelihood theories. The relation of the "operational logic" of an entity to the "predictive logic" of a likelihood theory is commented on in Sections 4.1 and 7.

Example 3.3. The unit lattice and statistical theories. The *unit lattice* is defined to be the likelihood poset $\mathscr{L}_U = ([0, 1], \leq, \sim)$, where $[0, 1]$ denotes the unit interval, $\leq$ is the natural ordering of the reals, and $\sim$ is the involution $\sim x = 1 - x$. A statistical theory may be characterized as a likelihood theory with predictive component $(\mathscr{L}_U, p)$, such that p satisfies equations (1).

Example 3.4. A symmetric construction for likelihood posets. A wide class of likelihood posets, including those of the preceding examples, may be obtained via a very simple construction. In particular, if σ is any symmetric binary relation on some set V, define $W \subseteq V$ to be *σ-closed* if $W = (W^\sigma)^\sigma$, where

$$W^\sigma := \{v \in V \mid v \sigma \omega \text{ for all } w \in W\} \tag{6}$$

It can then be shown (e.g., Birkhoff, 1967, Section V.7) that the triple $(\Sigma(V), \subseteq, \hat{\sigma})$ is a likelihood poset, where $\Sigma(V)$ denotes the set of σ-closed subsets of V, $\subseteq$ is the set-inclusion relation, and $\hat{\sigma}$ denotes the involution which maps W to W^σ. Moreover, the poset $(\Sigma(V), \subseteq)$ is a *complete lattice* [i.e., the least upper and greatest lower bounds with respect to $\subseteq$ exist for every subset of $\Sigma(V)$], and $\hat{\sigma}$ is *order-reversing* (i.e., $W_1 \subseteq W_2$ implies $W_2^\sigma \subseteq W_1^\sigma$).

The properties of completeness and order-reversal in fact fully characterize those likelihood posets which may be generated in this manner. For, if $\mathscr{L} = (L, \leq, \sim)$ satisfies these two properties, then it can be shown that $\mathscr{L}$ is isomorphic to $(\Sigma(L), \subseteq, \hat{\sigma})$ under the mapping $x \to \{\sim x\}^\sigma$, where σ denotes the symmetric relation $x \leq \sim y$. For the case of Examples 3.1–3.3, σ has the alternative form $x + y \leq 1$.

3.3. Discussion

Likelihood, as characterized in Section 3.1, is a much weaker notion than probability. In particular, there are no metric, well-ordering, or additivity requirements imposed.

A few remarks are in order concerning the possibility of testing the predictions of likelihood theories. For both *binary* and *ternary* likelihood theories (Examples 3.1 and 3.2), where 0, $\frac{1}{2}$, and 1 are taken to correspond to "impossible," "indeterminate," and "certain," respectively, predictions are indeed testable. In particular, the predictions $l(E_\alpha, s) = 0$ and $l(E_\alpha, s) = 1$

may be *refuted* by experiment (via verification of experimental propositions E_α and E_{α^c}, respectively, for state s), while the prediction $l(E_\alpha, s) = \frac{1}{2}$ may be *confirmed* by experiment (via verification of experimental propositions E_α and E_{α^c} on separate occasions, for state s).

The predictions of *statistical* theories are at least partially testable, via the naturally associated *ternary* likelihood theory (obtained by mapping probability x to likelihood $\frac{1}{2}$ for $0 < x < 1$). However, interpretational difficulties inherited from classical probability theory present problems with regard to complete testability. For example, thinking of probabilities as "limiting relative frequencies" implies that no measured relative frequency can actually determine a probability (since an experimenter only has access to a finite number of experimental results). Even if a statistical theory predicts the probability of a given relative frequency, this probability is not measurable itself as a relative frequency by the same argument. Jeffreys (1961, Chapter VIII) provides an excellent discussion of such problems, and concludes that probabilities should be interpreted as "reasonable degrees of belief." More pragmatically, a statistical theory may be viewed as providing a "well-ordered list of betting odds" (Lande, 1965, p. 140), or, in the terminology of de Finetti (1974), a "coherent prevision," for the possible outcomes of experiments. But how is an experimenter to judge objectively between the "coherent previsions" of two rival theories?

The characterization of likelihood theories here offers two possible alternatives in this regard. First, one could from the outset deal only with *ternary* likelihood theories, which are testable. In the context of this paper, it will be seen that this alternative suggests a special "order-determining" assumption for the states of a physical theory (see Section 5.3).

Second, one could decide to accept likelihood, rather than probability, as the appropriate primitive notion in the axiomatization of physical theories. This alternative will be adopted here, as it (i) provides a sufficient basis for defining and discussing "predictive logics" for physical theories; (ii) includes probability as a special case; and (iii) allows for the possibility of a future generalized characterization of "probability" which permits (objectively) testable predictions.

4. THE PREDICTIVE LOGIC OF A LIKELIHOOD THEORY

4.1. Abstract Propositions

The theoretical concepts of "position," "momentum," "mass," "spin," "charge," etc., which appear in current physical theories are abstracted from a range of empirical phenomena. In particular, propositions relating to such concepts (e.g., "the position is to the left of marker A") may be tested by many different experiments, and thus correspond to *equivalence classes* of experimental propositions.

The characterization of physical theories in this paper (Section 2) permits only *one* natural manner for the (nontrivial) construction of such classes: two experimental propositions of a physical theory are defined to be *equivalent* with respect to the theory when the predictions of the theory do not distinguish between them. The relation of equivalence will be denoted by $\equiv$, and for the case of likelihood theories (Section 3.1), is given by

$$E_\alpha \equiv F_\beta \quad \text{if and only if} \quad l(E_\alpha, s) = l(F_\beta, s) \quad \text{for all} \quad s \in S \tag{7}$$

Relation (7) implies that the experimental propositions of a likelihood theory may be represented up to equivalence by *mappings*, from the set of states S to the set of likelihoods L, where $E_\alpha \in P$ is represented by the mapping $\mathscr{E}_\alpha$, with

$$\mathscr{E}_\alpha(s) := l(E_\alpha, s) \tag{8}$$

These mappings will be called the *abstract* propositions of the theory, and the set of such propositions denoted by $\mathscr{P}$ [this improves on the notation of Hall (1988), where the distinction between P and $\mathscr{P}$ is made by context only]. The relation $E_\alpha \equiv F_\beta$ holds if and only if $\mathscr{E}_\alpha = \mathscr{F}_\beta$.

An abstract proposition $\mathscr{A} \in \mathscr{P}$ is *testable* via experiment $E \in X$ if $\mathscr{A} = \mathscr{E}_\alpha$ for some $\alpha \subseteq R(E)$, in which case $\mathscr{A}$ is verified by experiment E if the experimental proposition E_α is verified. In general, an abstract proposition $\mathscr{A} \in \mathscr{P}$ may be tested by a number of distinct experiments $E, F, G, \ldots$, provided only that $\mathscr{A} = \mathscr{E}_\alpha = \mathscr{F}_\beta = \mathscr{G}_\gamma = \ldots$ for suitable $\alpha, \beta, \gamma, \ldots$. The likelihood of $\mathscr{A}$ being verified, if tested for state $s \in S$, is given from (8) by $\mathscr{A}(s)$.

The natural formation of equivalence classes for experimental propositions, via definition (7), may be compared with the method of the "operational approach" to quantum mechanics. In this approach, the construction of such classes relies on the "common practice of outcome identification" (Randall and Foulis, 1979, p. 172), i.e., "once we have assembled those physical operations of concern to us in a particular effort, we are almost inevitably moved, either by custom or by a particular intent, to identify certain outcomes of different operations" (Randall and Foulis, 1979, p. 171). Representing this initial identification process by allowing result sets for distinct experiments to overlap (e.g., Foulis *et al.*, 1983, p. 814), we can formulate the "operational" construction here as follows [definitions are taken from Bennett and Foulis (1990), Section 3].

First, define $E_\alpha, F_\beta \in P$ to be *local complements* if $\alpha \cap \beta = \varnothing$ and $\alpha \cup \beta = R(G)$ for some $G \in X$. Second, define the *perspectivity* relation $E_\alpha \approx F_\beta$ to hold when E_α and F_β share a common local complement. Third, define the *implication* relation $E_\alpha \Rightarrow F_\beta$ to hold when there is some sequence $E_{\alpha_1}^{(1)}, \ldots, E_{\alpha_n}^{(n)} \in P$ such that $E_{\alpha_1}^{(1)} = E_\alpha$, $E_{\alpha_n}^{(n)} = F_\beta$, and either $\alpha_i \subseteq \alpha_{i+1}$ or $E_{\alpha_i}^{(i)} \approx E_{\alpha_{i+1}}^{(i+1)}$ for each $i = 1, 2, \ldots, n-1$. Finally, define E_α, F_β to be *operationally equivalent* if $E_\alpha \Rightarrow F_\beta$ and $F_\beta \Rightarrow E_\alpha$.

It may be checked that operational equivalence is indeed an equivalence relation on the set of experimental propositions. However, if *no* experimental outcomes are initially identified, then operational equivalence reduces to the trivial relation of identity. Hence, any "physics" arising from the operational construction is in fact contained in the initial, unspecified identification process. Further, for a given identification process there is no guarantee that predictions of a physical theory will respect operational equivalence.

The reliance on some *a priori* identification process does not affect the usefulness of the operational approach in "formulating a precision 'language' in which . . . theories can be expressed, compared, evaluated, and related to laboratory experiments" (Foulis and Randall, 1981, p. 9). The operational approach does, however, provide a physically ambiguous basis for an *axiomatic* approach to quantum theory, in that the all-important identification process is not specified. This contrasts with the framework of likelihood theories, in which the relation of "predictive" equivalence [relation (7)] uniquely and naturally specifies equivalence classes of experimental propositions.

4.2. Predictive Logics

The abstract propositions of a likelihood theory possess a natural logical structure. First, likelihood postulate (L1) of Section 3.1 implies the existence of a natural involution, $\sim$, on the set of abstract propositions, where $\sim\mathscr{A} \in \mathscr{P}$ is defined for each $\mathscr{A} \in \mathscr{P}$ by

$$(\sim\mathscr{A})(s) := \sim(\mathscr{A}(s)) \tag{9}$$

Second, likelihood postulate (L2) implies the existence of a natural partial ordering, $\leq$, on $\mathscr{P}$, defined by

$$\mathscr{A} \leq \mathscr{B} \qquad \text{if and only if} \quad \mathscr{A}(s) \leq \mathscr{B}(s) \quad \text{for all} \quad s \in S \tag{10}$$

The *predictive logic* of a likelihood theory is defined to be the involutive poset $(\mathscr{P}, \leq, \sim)$. Conditions (2a)–(2c) for likelihood theories may be rewritten as properties of the predictive logic:

$$\mathscr{E}_{\alpha^c} = \sim\mathscr{E}_\alpha \tag{11a}$$

$$\mathscr{E}_\alpha \leq \mathscr{E}_\beta \qquad \text{for} \quad \alpha \subseteq \beta \tag{11b}$$

$$\mathscr{E}_\varnothing = \mathbf{0} \tag{11c}$$

where the existence of the special abstract proposition **0** in (11c) follows from condition (2c). The abstract proposition $\sim\mathbf{0}$ will be denoted by **1**.

The conjunction, $\mathscr{A}\wedge\mathscr{B}$, and the disjunction, $\mathscr{A}\vee\mathscr{B}$, of two abstract propositions $\mathscr{A},\mathscr{B}\in\mathscr{P}$ may be implicitly defined in the usual manner as greatest lower and least upper bounds:

$$\mathscr{C}\leq\mathscr{A}\wedge\mathscr{B} \quad \text{if and only if} \quad \mathscr{C}\leq\mathscr{A}, \quad \mathscr{C}\leq\mathscr{B} \tag{12a}$$

$$\mathscr{A}\vee\mathscr{B}\leq\mathscr{C} \quad \text{if and only if} \quad \mathscr{A}\leq\mathscr{C}, \quad \mathscr{B}\leq\mathscr{C} \tag{12b}$$

Thus, $\mathscr{A}\wedge\mathscr{B}$ and $\mathscr{A}\vee\mathscr{B}$ are only well-defined when the corresponding bounds exist (though see Section 4.3 below for the case where the likelihood poset forms a complete lattice).

It follows from relations (12) that the connectives $\wedge$ and $\vee$ are idempotent, commutative, and associative. Moreover, interpreting the relation $\mathscr{A}\leq\mathscr{B}$ as "$\mathscr{A}$ is less likely than $\mathscr{B}$," it follows that $\mathscr{A}\wedge\mathscr{B}$ has the greatest likelihood of being verified in any state, of all those propositions less likely than both $\mathscr{A}$ and $\mathscr{B}$. Similarly, $\mathscr{A}\vee\mathscr{B}$ has the *least* likelihood of being verified in any state, of all propositions more likely than both $\mathscr{A}$ and $\mathscr{B}$.

The propositions $\mathscr{A}\wedge\mathscr{B}$ and $\mathscr{A}\vee\mathscr{B}$ (when they exist) are directly related by de Morgan's law in the case where the predictive logic has an order-reversing involution. In particular, the relations

$$\mathscr{A}\vee\mathscr{B}=\sim(\sim\mathscr{A}\wedge\sim\mathscr{B}) \qquad \text{for all} \quad \mathscr{A},\mathscr{B}\in\mathscr{P} \tag{13a}$$

$$\mathscr{A}\leq\mathscr{B} \quad \text{implies} \quad \sim\mathscr{B}\leq\sim\mathscr{A} \qquad \text{for all} \quad \mathscr{A},\mathscr{B}\in\mathscr{P} \tag{13b}$$

are equivalent. [To demonstrate this equivalence, note first that (13a) follows from (13b) using relations (12). Conversely, if (13a) holds, then $\mathscr{A}\leq\mathscr{B}$ implies $\mathscr{A}\vee\mathscr{B}=\mathscr{B}$ from (12b), and hence $\sim\mathscr{B}=\sim\mathscr{A}\wedge\sim\mathscr{B}$ from (13a). Substitution into (12a) with $\mathscr{C}=\sim\mathscr{B}$ yields (13b).] Note that a sufficient condition for (13b) [and hence (13a)] to hold follows from (9) and (10):

$$x\leq y \quad \text{implies} \quad \sim y\leq\sim x \qquad \text{for all} \quad x,y\in L \tag{14}$$

This condition is satisfied by all the examples in Section 3.2.

Conditions for the predictive logic of a likelihood theory to form either an orthomodular poset or a Boolean lattice are considered in Section 5. First, a generalization of the predictive logic is given for the case where the likelihood poset forms a complete lattice.

4.3. Potential Abstract Propositions

Conditions (2a)–(2c), or equivalently (11a)–(11c), imply that *every* abstract proposition $\mathscr{A}\in\mathscr{P}$ must satisfy

$$\mathbf{0}\leq\mathscr{A}\leq\mathbf{1} \tag{15}$$

where **1** denotes $\sim\mathbf{0}$. Generalizing from Hall (1988, Section 3), the set of *potential* abstract propositions $\mathscr{P}\mathscr{P}$ is then defined as the set of *all* mappings from S to L which satisfy condition (15) [extending definition (10) to such mappings]. Thus, $\mathscr{P}\subseteq\mathscr{P}\mathscr{P}$, and for a given set of states S, all elements of $\mathscr{P}\mathscr{P}$ are *a priori* candidates for abstract propositions.

In the case where $(L, \leq)$ forms a *complete lattice* (see Example 3.4), the "conjunction" and "disjunction" of an arbitrary set $\{A_i\}$ in $\mathscr{P}\mathscr{P}$ are respectively defined by

$$(\wedge_i \mathscr{A}_i)(s) := \sup\{\mathscr{C}(s) \mid \mathscr{C}\in\mathscr{P}, \mathscr{C}\leq \cap_i \mathscr{A}_i\} \tag{16a}$$

$$(\vee_i \mathscr{A}_i)(s) := \inf\{\mathscr{C}(s) \mid \mathscr{C}\in\mathscr{P}, \cup_i \mathscr{A}_i \leq \mathscr{C}\} \tag{16b}$$

where $\cap_i \mathscr{A}_i$, $\cup_i \mathscr{A}_i \in \mathscr{P}\mathscr{P}$ are given by

$$(\cap_i \mathscr{A}_i)(s) := \inf\{\mathscr{A}_i(s)\}, \qquad (\cup_i \mathscr{A}_i)(s) := \sup\{\mathscr{A}_i(s)\} \tag{16c}$$

and inf and sup denote least upper and greatest lower bounds, respectively, in $(L, \leq)$.

The above definitions directly generalize those in Hall (1988, Section 3) and satisfy similar properties. In particular, definitions (16a) and (16b) are consistent extensions of (12a) and (12b) and satisfy associativity and commutativity. The extended definitions confer the technical advantage that $\mathscr{A}\wedge\mathscr{B}$ and $\mathscr{A}\vee\mathscr{B}$ always exist as potential abstract propositions, if not as lower and upper bounds. Further, they allow the definition of the *closure*, $\bar{\mathscr{P}}$, of $\mathscr{P}$:

$$\bar{\mathscr{P}} := \{\mathscr{A}\in\mathscr{P}\mathscr{P} \mid \mathscr{A}\wedge\mathscr{A} = \mathscr{A}\vee\mathscr{A}\} \tag{17}$$

It can be shown that $\mathscr{P}\to\bar{\mathscr{P}}$ is a closure relation on $\mathscr{P}\mathscr{P}$ (i.e., $\mathscr{P}\subseteq\bar{\mathscr{P}}$, $\bar{\bar{\mathscr{P}}}=\bar{\mathscr{P}}$, $\mathscr{P}_1\subseteq\mathscr{P}_2$ implies $\bar{\mathscr{P}}_1\subseteq\bar{\mathscr{P}}_2$), and that $\bar{\mathscr{P}}$ is the maximal extension of $\mathscr{P}$ in $\mathscr{P}\mathscr{P}$ which preserves "joins" and "meets" [as defined in (16)].

5. AXIOMS FOR PREDICTIVE LOGICS

5.1. Introduction

The predictive logic of a likelihood theory is seen to be a very simple structure. The abstract propositions identify those classes of experimental propositions equivalent with respect to prediction, and have a natural partial-ordering and involution directly induced from the fundamental likelihood postulates (L1), (L2). No *a priori* identification process is necessary, as in the "operational approach" (Section 4.1).

The predictive logic reflects the relationship between the descriptive and predictive components of the theory. In particular, a change in either component will in general lead to a change in the predictive logic. The properties of the latter may therefore be used to characterize various classes of physical theories. For example, the predictive logics of classical "phase space" theories are always Boolean , while those of quantum "Hilbert space" theories are not [e.g., Examples 3.1 and 3.2 of Hall (1988)].

The characterization of possible theories of quantum phenomena in terms of their predictive logics may be considered a basic aim of the "quantum logic" approach to axiomatic quantum theory. Given that the successful building blocks of this approach are *orthomodular posets* (e.g., Varadarajan, 1985), it is important to determine suitable conditions for a predictive logic to form such a poset. Conditions deemed suitable are those which have a direct significance for the *physical properties* of abstract propositions, rather than conditions motivated on purely technical grounds.

For example, noting from (11a) that $\sim\mathscr{A}\in\mathscr{P}$ and $\sim(\sim\mathscr{A})=\mathscr{A}$ for all $\mathscr{A}\in\mathscr{P}$, it follows that the predictive logic of a likelihood theory forms an orthomodular poset if and only if:

$$\mathscr{A}\wedge\sim\mathscr{A}=\mathbf{0} \tag{18a}$$

and

$$\mathscr{A}\leq\mathscr{B}\quad\text{implies}\qquad\text{(i)}\quad \sim\mathscr{B}\leq\sim\mathscr{A} \tag{18b}$$

$$\text{(ii)}\quad \sim\mathscr{A}\wedge\mathscr{B}\in\mathscr{P} \tag{18c}$$

$$\text{(iii)}\quad \mathscr{A}\vee(\sim\mathscr{A}\wedge\mathscr{B})=\mathscr{B} \tag{18d}$$

for all $\mathscr{A}$, $\mathscr{B}\in\mathscr{P}$. Conditions (18a) and (18b) ensure that $\sim$ is an orthocomplementation; conditions (18c) and (18d) then ensure orthocompleteness and orthomodularity, respectively (e.g., Beltrametti and Cassinelli, 1981, Chapter 10). However, these conditions are *not* suitable as *physical* axioms for orthomodularity, as they have no direct interpretation in terms of the physical properties of abstract propositions.

In contrast, the following section provides two simple axioms for orthomodularity, which relate the structure of the predictive logic to the physical property of *joint testability* for abstract propositions. Moreover, strengthening one of these axioms to the postulate that any two abstract propositions are jointly testable implies that the predictive logic forms a Boolean lattice. A possible third axiom is discussed in Section 5.3.

5.2. Orthomodular Posets and Boolean Lattices

Consider the case where two abstract propositions $\mathscr{A}$, $\mathscr{B}\in\mathscr{P}$ are *jointly testable*, i.e., $\mathscr{A}=\mathscr{E}_\alpha$ and $\mathscr{B}=\mathscr{E}_\beta$ for some experiment $E\in X$ and subsets $\alpha,\beta\subseteq R(E)$. Now, since $r\in\alpha\cap\beta$ if and only if $r\in\alpha$ and $r\in\beta$, then $E_{\alpha\cap\beta}\in P$ is the experimental proposition which is verified if and only if E_α and E_β

are verified. It follows (Section 4.1) that the abstract proposition $\mathscr{E}_{\alpha\cap\beta}$ corresponds to the joint verification of $\mathscr{A}$ and $\mathscr{B}$, via experiment E. The following axiom for predictive logics identifies the *conjunction* of such $\mathscr{A}$ and $\mathscr{B}$ with *joint verification*:

Axiom 1.

$$\mathscr{E}_\alpha \wedge \mathscr{E}_\beta = \mathscr{E}_{\alpha\cap\beta} \qquad \text{for all} \quad E \in X, \quad \alpha, \beta \subseteq R(E) \tag{19}$$

Axiom 1 equivalently identifies the *likelihood of joint verification* $\mathscr{E}_{\alpha\cap\beta}(s)$ (for abstract propositions $\mathscr{E}_\alpha$, $\mathscr{E}_\beta$ on state s) with the corresponding *joint likelihood* $(\mathscr{E}_\alpha \wedge \mathscr{E}_\beta)(s)$. It can for the purposes of this paper be replaced by the weaker condition

$$\mathscr{E}_\alpha \vee \mathscr{E}_\beta = \mathscr{E}_{\alpha\cup\beta} \qquad \text{whenever} \quad \alpha \cap \beta = \varnothing \tag{19a}$$

However, no physical insight appears to be gained by doing so.

A second axiom for predictive logics is motivated by the argument that if $(\mathscr{P}, \leq, \sim)$ has some fundamental physical significance, then, in particular, the relation $\mathscr{A} \leq \mathscr{B}$ should indicate some physical connection between $\mathscr{A}$ and $\mathscr{B}$. But the *simplest* physical relationship between two abstract propositions is that they are jointly testable, suggesting:

Axiom 2.

$$\mathscr{A} \leq \mathscr{B} \text{ implies } \mathscr{A} \text{ and } \mathscr{B} \text{ are jointly testable} \tag{20}$$

Axioms 1 and 2 together yield the following characterization of $\leq$:

Lemma. Under Axioms 1 and 2, the relation $\mathscr{A} \leq \mathscr{B}$ holds if and only if there exist $E \in X$ and $\alpha, \beta \subseteq R(E)$ such that (i) $\mathscr{A} = \mathscr{E}_\alpha$, $\mathscr{B} = \mathscr{E}_\beta$; and (ii) $\alpha \subseteq \beta$.

Proof. Suppose first that $\mathscr{A} \leq \mathscr{B}$. Then (i) follows directly from Axiom 2. Hence $\mathscr{A} \wedge \mathscr{B} = \mathscr{A} = \mathscr{E}_{\alpha\cap\beta}$ from definition (12a) and Axiom 1, so that α may be replaced by $\alpha \cap \beta$. But $\alpha \cap \beta \subseteq \beta$, proving (ii). Conversely, if (i) and (ii) hold, then $\mathscr{A} \leq \mathscr{B}$ follows immediately from (11b). ■

This lemma has the "logical" consequence that if $\mathscr{A} \leq \mathscr{B}$, then $\mathscr{A}$ and $\mathscr{B}$ may be jointly tested such that *$\mathscr{A}$ is verified only if $\mathscr{B}$ is verified.*

Axioms 1 and 2 relate $\wedge$ and $\leq$ in a very simple manner to the physical property of joint testability [and may indeed be considered as necessary if one wishes to interpret $(\mathscr{P}, \leq, \sim)$ as a propositional calculus]. That any predictive logic satisfying Axioms 1 and 2 forms an orthomodular poset is the content of the following theorem:

Theorem 1. Axioms 1 and 2 provide sufficient conditions for the predictive logic of a likelihood theory to form an orthomodular poset.

Proof. It must be demonstrated that conditions (18) follow from (19) and (20). First, noting from Section 4.1 that every $\mathscr{A} \in \mathscr{P}$ has the form $\mathscr{A} = \mathscr{E}_\alpha$, it follows from (11a), (11c) and (19) that

$$\mathscr{A} \wedge \sim\mathscr{A} = \mathscr{E}_\alpha \wedge \mathscr{E}_{\alpha^c} = \mathscr{E}_{\alpha \cap \alpha^c} = \mathscr{E}_\varnothing = \mathbf{0}$$

proving (18a). Second, suppose now that $\mathscr{A} \leq \mathscr{B}$ for some $\mathscr{A}, \mathscr{B} \in \mathscr{P}$. Then by the above lemma, one may write $\mathscr{A} = \mathscr{E}_\alpha$, $\mathscr{B} = \mathscr{E}_\beta$ with $\alpha \subseteq \beta$. Hence $\beta^c \subseteq \alpha^c$, so that $\sim\mathscr{B} \leq \sim\mathscr{A}$ from (11a) and (11b), proving (18b). Further, from (11a) and (19), $\sim\mathscr{A} \wedge \mathscr{B} = \mathscr{E}_{\alpha^c \cap \beta} \in \mathscr{P}$, proving (18c). Finally, since (13a) and (18b) are equivalent (Section 4.2), it follows from (11a), (19), and $\alpha \subseteq \beta$ that

$$\mathscr{A} \vee (\sim\mathscr{A} \vee \mathscr{B}) = \sim(\sim\mathscr{A} \wedge \sim(\sim\mathscr{A} \wedge \mathscr{B})) = \mathscr{E}_{(\alpha^c \cap (\alpha^c \cap \beta)^c)^c} = \mathscr{E}_\beta = \mathscr{B}$$

proving (18d). ■

For the case of *statistical* theories (Example 3.3 of Section 3), Theorem 1 admits an important corollary concerning the existence of "probability measures." A mapping $m: \mathscr{P} \to [0, 1]$ for an orthomodular poset $(\mathscr{P}, \leq, \sim)$ is a (finitely-additive) *probability measure* if the conditions

$$m(\mathbf{1}) = 1 \tag{21a}$$

$$m(\mathscr{A} \vee \mathscr{B}) = m(\mathscr{A}) + m(\mathscr{B}) \qquad \text{for} \quad \mathscr{A} \leq \sim\mathscr{B} \tag{21b}$$

are satisfied for all $\mathscr{A}, \mathscr{B} \in \mathscr{P}$. Further, a set M of probability measures is said to be *order-determining*, or *full*, on $(\mathscr{P}, \leq, \sim)$ if

$$m(\mathscr{A}) \leq m(\mathscr{B}) \quad \text{for all} \quad m \in M \qquad \text{implies} \qquad \mathscr{A} \leq \mathscr{B} \tag{22}$$

We have the following result:

Corollary. Axioms 1 and 2 provide sufficient conditions for the predictive logic of a statistical theory to admit an order-determining set of probability measures, $M = \{m_s \mid s \in S\}$, where $m_s: \mathscr{P} \to [0, 1]$ is defined for each $s \in S$ by

$$m_s(\mathscr{A}) := \mathscr{A}(s) \tag{23}$$

Proof. First, $(\mathscr{P}, \leq, \sim)$ is an orthomodular poset from Theorem 1, and from definitions (8) and (23):

$$m_s(\mathscr{E}_\alpha) = p(E_\alpha, s) \tag{24}$$

where the likelihood mapping p satisfies conditions (1). Now, from (11a) and (11c) one has $\mathbf{1} = \sim\mathbf{0} = \sim\mathscr{E}_\varnothing = \mathscr{E}_{R(E)}$, and hence (1a) and (24) imply that

(i) $\quad m_s(\mathbf{1}) = p(E_{R(E)}, s) = 1$

Further, if $\mathscr{A} \leq \sim\mathscr{B}$, then the preceding lemma and property (11a) imply that $\mathscr{A} = \mathscr{E}_\alpha$, $\mathscr{B} = \mathscr{E}_\beta$ with $\alpha \subseteq \beta^c$, i.e., $\alpha \cap \beta = \varnothing$. Therefore $\mathscr{A} \vee \mathscr{B} = \mathscr{E}_{\alpha \cup \beta}$, using (11a), (18b), and Axiom 1, and hence (1b) and (24) imply that

$$\text{(ii)} \qquad m_s(\mathscr{A} \vee \mathscr{B}) = p(E_{\alpha \cup \beta}, s) = p(E_\alpha, s) + p(E_\beta, s)$$
$$= m_s(\mathscr{E}_\alpha) + m_s(\mathscr{E}_\beta)$$
$$= m_s(\mathscr{A}) + m_s(\mathscr{B})$$

Conditions (21) follow from (i) and (ii) above, i.e., m_s is a probability measure on $(\mathscr{P}, \leq, \sim)$. Finally, (22) follows immediately from (10), (23), and (24), so that M is order-determining on $(\mathscr{P}, \leq, \sim)$. ■

Theorem 1 is of significance to the quantum logic approach, in that orthomodularity follows from rather simpler axioms than have been used hitherto (see Section 7). The following theorem demonstrates further that the predictive logic of a likelihood lattice is *Boolean* under a natural strengthening of Axiom 2.

Theorem 2. Axiom 1, and the assumption that any two abstract propositions may be jointly tested, provide sufficient conditions for the predictive logic of a likelihood theory to form a Boolean lattice.

Proof. The assumption of joint testability implies that any two abstract propositions $\mathscr{A}, \mathscr{B} \in \mathscr{P}$ have the form $\mathscr{A} = \mathscr{E}_\alpha$, $\mathscr{B} = \mathscr{E}_\beta$ for some $E \in X$ and $\alpha, \beta \subseteq R(E)$. Moreover, noting that this assumption is stronger than Axiom 2, it follows from Theorem 1 that conditions (18) hold. Then, using (11a), (18b), and Axiom 1,

$$\text{(i)} \qquad \mathscr{A} \vee \mathscr{B} = \sim(\sim\mathscr{A} \wedge \sim\mathscr{B}) = \mathscr{E}_{(\alpha^c \cap \beta^c)^c} = \mathscr{E}_{\alpha \cup \beta} \in \mathscr{P}$$

and

$$\text{(ii)} \qquad (\mathscr{A} \wedge \mathscr{B}) \vee (\sim\mathscr{A} \wedge \mathscr{B}) = \sim(\sim(\mathscr{A} \wedge \mathscr{B}) \wedge \sim(\sim\mathscr{A} \wedge \mathscr{B}))$$
$$= \mathscr{E}_{((\alpha \cap \beta)^c \cap (\alpha^c \cap \beta)^c)^c} = \mathscr{E}_\beta = \mathscr{B}$$

Also, from definition (12b) one has

$$\text{(iii)} \qquad \mathscr{A} \vee (\mathscr{B} \vee \mathscr{C}) = (\mathscr{A} \vee \mathscr{B}) \vee \mathscr{C}$$

and

$$\text{(iv)} \qquad \mathscr{A} \vee \mathscr{B} = \mathscr{B} \vee \mathscr{A}$$

But a theorem due to Huntingdon (1933; see also Birkhoff, 1967, Section 10) states that properties (i)–(iv) above are necessary and sufficient for $(\mathscr{P}, \leq, \sim)$ to form a Boolean lattice. ■

5.3. Ternary Completeness

In general, an orthomodular poset need not admit *any* probability measures (Greechie, 1971). However, the corollary to Theorem 1 demonstrates that Axioms 1 and 2 are sufficient for the predictive logic of a statistical theory to indeed admit the existence of an *order-determining* set of such measures.

A more general order-determining property may in fact be reasonably postulated for the states of a likelihood theory. The discussion in Section 3.3 suggested that problems regarding the testing of likelihood predictions, including statistical predictions, could be avoided by dealing with *ternary* likelihood theories. This essentially corresponds to replacing the predictive logic $(\mathscr{P}, \leq, \sim)$ of a likelihood theory with an associated ternary predictive logic $(\mathscr{P}_T, \leq, \sim)$, where for $\mathscr{A} \in \mathscr{P}$ one defines $\mathscr{A}_T \in \mathscr{P}_T$ by

$$\mathscr{A}_T(s) := \begin{cases} 1, & \mathscr{A}(s) = \mathbf{1}(s) \\ 0, & \mathscr{A}(s) = \mathbf{0}(s) \\ \frac{1}{2}, & \text{otherwise} \end{cases} \tag{25}$$

Thus $\mathscr{A}_T$ maps S to L_3 (Example 3.2 of Section 3), and $\mathscr{A}_T(s)$ characterizes whether $\mathscr{A}$ is "certain," "impossible," or "indeterminate" for state s.

From (25) it follows that $(\sim\mathscr{A})_T = \sim(\mathscr{A}_T)$, and that $\mathscr{A}_T \leq \mathscr{B}_T$ whenever $\mathscr{A} \leq \mathscr{B}$. Hence the two predictive logics $(\mathscr{P}, \leq, \sim)$ and $(\mathscr{P}_T, \leq, \sim)$ will be *isomorphic* if and only if

$$\mathscr{A}_T \leq \mathscr{B}_T \quad \text{implies} \quad \mathscr{A} \leq \mathscr{B} \tag{26}$$

In such a case, $(\mathscr{P}, \leq, \sim)$ may then be recovered from the fundamental (and testable) *ternary* predictions of the theory, motivating (26) as a third axiom for likelihood theories. Condition (26) will be referred to as *ternary completeness*.

It follows from definition (25) that $\mathscr{A}_T \leq \mathscr{B}_T$ if and only if $S_\mathscr{A} \subseteq S_\mathscr{B}$ and $S_{\sim\mathscr{B}} \subseteq S_{\sim\mathscr{A}}$, where $S_\mathscr{A}$ denotes the subset of states for which $\mathscr{A} \in \mathscr{P}$ is "certain" to be verified, i.e.,

$$S_\mathscr{A} := \{s \in S \mid \mathscr{A}(s) = \mathbf{1}(s)\} \tag{27}$$

Hence condition (26) has the equivalent form

$$S_\mathscr{A} \subseteq S_\mathscr{B} \quad \text{and} \quad S_{\sim\mathscr{B}} \subseteq S_{\sim\mathscr{A}} \quad \text{implies} \quad \mathscr{A} \leq \mathscr{B} \tag{28}$$

In this form it is clearly seen to be weaker than the commonly considered *strongly* order-determining condition (Beltrametti and Cassinelli, 1971, Chapter 11)

$$S_{\mathscr{A}} \subseteq S_{\mathscr{B}} \quad \text{implies} \quad \mathscr{A} \leq \mathscr{B} \tag{29}$$

A likelihood theory satisfying ternary completeness may alternatively, in analogy to condition (29), be said to have a *ternary* order-determining set of states.

It is well known that quantum "Hilbert space" theories have a strongly order-determining set of states. Yet Greechie (1978) has given examples of orthomodular posets which admit an order-determining set of probability measures, but *not* a strongly order-determining set of states. This implies the existence of statistical theories satisfying Axioms 1 and 2 which cannot be embedded in the Hilbert space formalism. It would therefore be of interest to determine whether Axioms 1 and 2 provide sufficient conditions for a statistical theory with a *ternary* order-determining set of states to in fact admit a strongly order-determining set. Greechie's examples could then be bypassed by the assumption of ternary completeness as a third axiom for likelihood theories.

6. DYNAMICAL ASPECTS

The predictive logic of a likelihood theory is formed in a natural manner from the entire set of predictions made by the theory. Its invariance properties should therefore be expected to reflect generic features of the class of systems described by the theory.

For example, if this class is invariant relative to the set of inertial frames in Minkowski space, then the predictive logics relative to such frames should be connected by a set of isomorphisms which form a representation of the Lorentz group. Similarly, if the class of systems is invariant relative to the time shown on some "clock," then the predictive logics relative to different clock times should be isomorphic.

In general, the invariance or symmetry properties of a predictive logic $(\mathscr{P}, \leq, \sim)$ will be characterized by the group $\mathbf{H}$ of logic-preserving automorphisms on $\mathscr{P}$:

$$\mathbf{H} := \{g \in \mathrm{Aut}(\mathscr{P}) \mid g(\sim\mathscr{A}) = \sim g(\mathscr{A});\, g(\mathscr{A}) \leq g(\mathscr{B}) \text{ for } \mathscr{A} \leq \mathscr{B}\} \tag{30}$$

This group will be called the *logical symmetry group* of the theory. Any logic-preserving time evolution $\mathscr{A} \to \mathscr{A}_t$ for the abstract propositions of the theory must be described by a one-parameter family $\{g_t\}$ in $\mathbf{H}$, with $\mathscr{A}_t = g_t(\mathscr{A})$.

The logical symmetry group of a likelihood theory has an important subgroup $\mathbf{S}$ consisting of those elements in $\mathbf{H}$ which may be implemented as automorphisms on the set of states of the theory, i.e.,

$$\mathbf{S} := \{g \in \mathbf{H} \mid \exists \hat{g} \in \mathrm{Aut}(S) \text{ with } g(\mathscr{A}) = \mathscr{A} \circ \hat{g}\} \tag{31}$$

In particular, if $\{g_t\}$ is a one-parameter family in **S**, then the evolution of likelihoods under $\{g_t\}$ can be equivalently represented either by the evolution of propositions, $\mathscr{A}_t = g_t(\mathscr{A})$, or by the evolution of states, $s_t = \hat{g}_t(s)$. In either case, likelihoods evolve as

$$[\mathscr{A}(s)]_t = \mathscr{A}_t(s) = \mathscr{A}(s_t) \tag{32}$$

The subgroup **S** will be called the *state symmetry* group of the theory.

In standard quantum mechanics, **H** corresponds to the Heisenberg picture, in which propositions (represented by projection operators) evolve with time, while **S** corresponds to the equivalent Schrödinger picture, in which states (represented by unit vectors or density operators) evolve with time. However, in the more general framework of likelihood theories an equivalent "Schrödinger picture" cannot always be defined: **S** is typically a *proper* subgroup of **H**. A simple "discrete" example is provided by a deterministic theory (Example 3.1 of Section 3) with exactly four states, and a set of eight abstract propositions defined by

$$\mathscr{P} = \left\{ \mathscr{A} : S \to \{0, 1\} \,\middle|\, \sum_{s \in S} \mathscr{A}(s) \text{ is even} \right\} \tag{33}$$

It can be checked that in this case **H** has $6 \times 4 \times 2 = 48$ elements, which permute the six nontrivial abstract propositions while preserving involution, while **S** has only $4! = 24$ elements, corresponding to the permutation group of order 4.

This primary significance of the "Heisenberg picture" over the "Schrödinger picture" for likelihood theories follows from the lack of any assumed structure for the set of states S (such as convexity), and is in contrast to the approach discussed by Beltrametti and Cassinelli (1981, Chapter 23), where the evolution of states rather than propositions is given the greater fundamental significance. It is of interest to note here the view of Dirac (1966, Chapter 1) that certain difficulties in quantum field theory may be resolved by in fact discarding the Schrödinger picture in favor of a (nonequivalent) Heisenberg picture of evolution.

7. CONCLUDING REMARKS

The existence of inherent logical structures for classical and quantum theories was first emphasized by Birkhoff and von Neumann (1936). The existence of such structures for a much wider class of physical theories has been demonstrated here, based on the primitive notion of "likelihood."

In this paper, probability and logic are relatively independent notions, though underpinned by a common concept (likelihood). Thus, probability is regarded as a special type of likelihood (Example 3.3 of Section 3), while logical structures are inevitable and natural features of physical theories which predict likelihoods (Section 4). This contrasts with the usual tradition of "logic first, probability second' in both classical and quantum probability theory, where a logical structure is typically postulated *a priori* for the propositions under consideration, and probabilities are then defined relative to this structure (e.g., Kolmogorov, 1950; Jeffreys, 1961; Cox, 1961; Beltrametti and Cassinelli, 1981; Pitowsky, 1989).

One of the main problems in the quantum logic approach to axiomatic quantum theory is to provide a sufficiently convincing setting or derivation for orthomodularity. For example, Maczynski (1974) relies on the additivity of probabilities and a technical "orthogonality postulate"; Marlow (1978) both on the additivity of probabilities and the *a priori* existence of an $\wedge$ operation; Foulis *et al.* (1983) on an *a priori* identification process for experimental outcomes and a technical "orthocoherence" condition; and Ivanov (1991) on a technical "commutativity" condition and the *a priori* existence of an ortholattice.

In contrast, Theorem 1 of Section 5.2 provides a path to orthomodularity based on the primitive notions of likelihood and joint testability. In particular, the existence of a predictive logic follows for *any* theory which predicts likelihoods (Section 4). Axioms 1 and 2 of Section 5.2 then provide sufficient conditions for this predictive logic to form an orthomodular poset. There are no additivity, existence, or purely technical assumptions made, as in the works mentioned above. Rather, Axioms 1 and 2 provide direct physical interpretations for $\wedge$ and $\leq$, related to joint testability, which are in fact implicitly assumed in most versions of the quantum logic approach.

In analogy to Theorem 1, discussed above, Theorem 2 of Section 5.2 provides a remarkably simple path to Boolean lattices, based on Axiom 1 plus the assumption of joint testability for any two abstract propositions. It is of technical interest to note that both theorems rely on conditions involving only *pairs* of propositions.

Finally, Example 3.2 of Section 3 indicates the close relationship of ternary likelihood theories with the "entities" defined by Foulis *et al.* (1983). The latter represent states as sets of possible outcomes (Example 3.2), and obtain an "operational logic" for an entity based on an *a priori* identification process for experimental outcomes (Section 4.1). In contrast, the predictive logic of a ternary likelihood theory is obtained simply as a particular case of likelihood theories in general (Section 4), with *no* assumptions necessary concerning the representation of states or the identification of outcomes belonging to distinct experiments. The framework of likelihood theories thus appears conceptually simpler than that of "entities."

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